# Black-Scholes Approximation of Warrant Prices: Slight Return in a Low Interest Rate Environment

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**Abstract**

The objective of this paper is to emphasize the differences between a call and a warrant as well as the different valuation methods of warrants which have been introduced in the financial literature. For the sake of simplicity and applicability, we only consider a debt-free equity-financed firm. More recently a formal distinction between structural and reduced form pricing models has been introduced. This distinction is important whether one wishes to price a new warrant issue or outstanding warrants. If we are interested in pricing a new issue of warrants, e.g. in the context of a management incentive package, one has to rely on a structural model. However most of practitioners use the simple Black-Scholes formula. In this context, we analyze the accuracy of the approximation of the "true" price of a warrant by the Black-Scholes formula. We show that in the current low interest rate environment, the quality of the approximation deteriorates and the sensitivity of this approximation to the volatility estimate increases.

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# 1 Introduction

Warrants are long term call options to buy the firm's own shares and are issued by the firm itself. Because of this latter characteristic, warrants are more complex to value than exchange-traded options.

As in credit risk studies[1], warrant pricing models fall into either the "structural" or the "reduced form" category. Strictly speaking, this distinction has been first put forward quite recently in the literature on warrant pricing by Jarrow and Trautmann (2011). But the works of Bensoussan, Crouhy and Galai (1994, 1995a and 1995b) already made this distinction without using the same terminology. The starting point for the analysis of a structural model is the evolution of structural variables of firms, such as the value of assets and debt, to determine among other things the value of warrants. These models deal directly with the dilution effect, that is the increase in shares if warrants are exercised. To the contrary, a reduced-form model starts with the assumptions about the stock price process and the outstanding warrants, meaning that they have already been issued (see Jarrow and Trautmann, 2011).

The distinction between these two types of models shows up particularly useful when we want to distinguish the valuation of an outstanding warrant from that of a warrant to be issued. A structural model, based on the value and on the volatility of the firm's assets, is well designed to price a new issue of warrant because in this case the stock price does not already reflect the fact that the warrant exists. As first shown by Crouhy and Galai (1991b) and Handley (2002) and then developed in more depth by Jarrow and Trautmann (2011), a reduced form model is more appropriate for outstanding warrant because in a efficient market, "the stock price of the underlying firm conditionally reflects dilution at all times following announcement of a warrant issue". Therefore, "pricing a warrant as a call option on the stock of the underlying firm does not actually ignore dilution because dilution already should be reflected in the underlying stock price before expiration of the warrant".

In the present paper, we don't address the issue of possible sequential optimal exercise of American warrants before maturity. The treatment of this issue is available in Chapter 19 of Ingersoll (1987). In addition, Koziol (2006) addresses extensively the issue of optimal exercise strategies for warrants issued by levered firms. Instead, we make here the assumption that American warrants are exercised as a block at maturity. Thus, we suppose that no dividends are paid (see A2 Schulz and Trautmann 1994).

Another point that needs to be addressed has to do with the cash received by the firm from the sale of warrants. The objective here is to ensure that these cash flows do not modify the rate of return and the risk of the firm's assets. Several possibilities are available : the issue proceeds are immediately distributed as cash dividends to shareholders, alternatively the firm can repurchase its stocks or use the proceeds to invest in a scale expansion of the firm. Following Galai and Schneller (1978), we retain the cash dividends assumption[2].

[1] See *e.g.* Jarrow and Protter (2004) for an analysis of this distinction in credit risk models.

[2] If one of the other two cases is retained, some of the relationships established below must be adjusted but not the warrant valuation formula.

When a new issue of warrants has to be priced, the issuer need to know which model he can use and, if so, which approximation is relevant. It is particularly useful in the context, for example, of warrants as support for incentive management programme. Such warrants are usually long term out-of-the-money warrants. On the contrary when dealing with outstanding warrants, the price of the underlying security already reflects the dilution. In this paper we are primarily interested in the pricing of new warrant issues in the simple case of a no-debt firm and the type of approximate formula that can be used by an issuer.

The remaining parts of the paper are organized as follows. In section 2, the warrant mechanisms are recalled. In section 3, the different warrant pricing models derived in the literature are presented and discussed. The quality of the approximation of the price of a warrant by a simple application of the Black-Scholes model is assessed in Section 4, with an emphasis on these results in a low interest rate environment. Section 5 summarizes the conclusion of this article.

## 2 Warrants mechanisms

In this section, we follow the standard presentation of Galai and Schneller (1978). A warrant is a call option written by a firm on its own common stock. As such, it is the right to buy a share[3] of the firm at a certain price (exercise price) during a given time period. If a warrant is exercised, new shares are issued by the firm, and the cash payment that is made by warrant buyers may or may not increase the assets of the firm depending of the scenario we are considering[4].

While the call option is issued by an outside business organization such as the CBOE, the warrant is issued by the firm and its proceeds are a part of the firm's equity. If a warrant is exercised, it increases the number of outstanding shares of the firm and thus dilutes the equity of its stockholders.

For comparison purpose, we analyze the same firm, *i.e.* a firm with same value of assets, but with two different capital structures: a pure equity financed firm and a firm financed both by equity and warrants. Moreover, we make the unrealistic assumptions that the firm has no debt. For the introduction of debt into the analysis of warrant pricing, we refer to Crouhy and Galai (l994), Abínzano and Navas (2013) and Simonato (2015). The firm operates over the time period $[t, T]$, at time $t$ warrants are issued with expiration at time $T$. The time $t$ price of a warrant is denoted by $W_t$. The pure equity firm has $N$ shares of stock outstanding at time $t$ and no debt. The firm's assets value at time $T$ is denoted by $\tilde{V}_T$. The value of this firm at time $t$ is denoted by $V_t^*$ with $V_t^* = NS_t^*$. The features of this firm are displayed in Table 1 first three rows. The price of a share at time $T$ is $\tilde{S}_T^*$ and it is equal to $\tilde{V}_T/N$. For the firm which plans to issue $M$ European warrants with an exercise price equal to $K$, some additional assumptions are in order. As already said, it is assumed that the proceeds received from the warrant buyers are distributed as cash dividends to

[3] In order to simplify the exposition and without loss of generality, we assume a conversion ratio of 1:1. That is, a warrant gives the right to buy a share.

[4] Note that the scenario considered has no effect on the wealth of the shareholders nor on the value of the warrant. See below for more details.

the existing shareholders[5].

If at time $T$ the warrants are exercised, the firm will be worth $\tilde{V}_T + MK$ and there are $(N+M)$ shares of stock outstanding. The price of share is now, $\tilde{S}_T = \frac{\tilde{V}_T+MK}{M+N}$. Thus, each warrant is exercised if :

$$\begin{aligned}\tilde{S}_T = \frac{\tilde{V}_T + MK}{N+M} &> K \\ \Leftrightarrow \frac{\tilde{V}_T - NK}{N+M} &> 0 \\ \Leftrightarrow \frac{N}{N+M}\left(\tilde{S}^*_T - K\right) &> 0 \end{aligned} \tag{1}$$

Finally, the warrants is exercised if $\tilde{S}^*_T > K$.

Table 1 shows that on the liability side of the balance sheet, the two firms have the same equity value at time $T$, $\tilde{V}_T$ and thus it must also be true at time $t$, $E_t = NS_t + MW_t = NS^*_t = V^*_t$. Thus, initial shareholders wealth is not affected by the issuance of warrants. On the assets side[6], $F_t = V^*_t + e^{-r\tau}\mathbb{E}^{\mathbb{Q}}_t[\mathbf{1}_{\tilde{S}^*_T>K}MK]$, that is the value of the firm with warrants is higher than the value of the pure equity firm. At first glance, this seems to contradict Modigliani-Miller's Proposition I (1958) which states that in perfect capital markets, the value of a firm is determined by the value of its assets and is independent of its capital structure, *i.e.* how the firm finances its investments. But what matters is shareholders wealth, which is not affected. Moreover, because we take into account the present value of the additional expected cash flow arising at time $T$ due to the potential exercise of the warrant, we must also take into account that the number of shareholders at time $t$ is also expected to be higher than $N$, so that equality on both sides of the balance sheet is restored.

The price of a share of the firm with warrants is lower than the price of a pure equity firm: $S_t = S^*_t - (M/N)\,W_t$. This is due to the cash payment of $MW_t$ and again reflects expectations of the dilution that could occur at time $T$.

# 3 Valuation of Warrants

Three general approaches to valuing warrants have been introduced in the academic literature (Black and Scholes 1973, Galai, and Schneller 1978 and Handley 2002) :

1. First, warrants are analyzed as call options on the value of the firm (*i.e.*, value of its shares of common stocks and its warrants),

2. Second, warrants are analyzed as call options on the stock of an otherwise identical all-equity firm[7],

[5] In this case, the price of the share must adjust downward because some money went out of the firm while at the same time the equity increases at time $t$, conditionally on the possibility of warrant's exercises at time $T$, see Handley (2002).

[6] $\mathbb{Q}$ denotes the risk neutral probability measure such that the price of each risky cash flow whose value depends on the only source of risk, *i.e.* the firm's assets value which is supposed to follow a geometric Brownian motion, is equal to the discounted expectation of its future value under this measure. The symbol $\mathbf{1}_{\tilde{S}^*_T>K}$ denotes the indicator function which is equal to one if $\tilde{S}^*_T > K$ and to zero if $\tilde{S}^*_T \leq K$. The risk-free interest rate is $r$ and the time to expiration of the warrant is $\tau = T - t$.

[7] An identical firm is a firm with the same assets value, $V$.

Table 1: Shares, Warrants and Firm Values

| | $t$ | $T$ | |
|---|---|---|---|
| | | $\tilde{S}_T^* \leq K$ | $\tilde{S}_T^* > K$ |
| 1) Firm with no warrants | | | |
| (a) Number of shares | $N$ | $N$ | $N$ |
| (b) Value of a share | $S_t^*$ | $\tilde{S}_T^* = \tilde{V}_T/N$ | $\tilde{S}_T^* = \tilde{V}_T/N$ |
| (c) Value of firm | $V_t^* = NS_t^*$ | $N\tilde{S}_T^* = \tilde{V}_T$ | $N\tilde{S}_T^* = \tilde{V}_T$ |
| 2) Firm with warrants | | | |
| (a) Number of shares | $N$ | $N$ | $N + M$ |
| (b) Value of a warrant | $W_t$ | 0 | $\frac{\tilde{V}_T+MK}{M+N} - K = \frac{N}{M+N}(\tilde{S}_T^* - K)$ |
| (c) Value of a share | $S_t$ | $\tilde{S}_T = \tilde{S}_T^*$ | $\tilde{S}_T = \frac{\tilde{V}_T+MK}{M+N}$ |
| (d) Value of firm's equity | $E_t = NS_t + MW_t = V_t^* \equiv V_t$ | $N\tilde{S}_T = \tilde{V}_T$ | $N\tilde{S}_T + M(\tilde{S}_T - K) = \tilde{V}_T$ |
| (e) Value of firm | $F_t = V_t^* + e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}[\mathbf{1}_{\tilde{S}_T^*>K} MK]$ | $\tilde{V}_T$ | $\tilde{V}_T + MK$ |

3. Third, warrants are analyzed as call options on the stock of the underlying firm.

These three approaches are all valid but in different contexts. They are general in the sense that they don't depend on a specific dynamics for the underlying security, either the value of the firm or the value of the stock.

As already mentioned, a more recent and distinct typology classifies the warrant pricing models according to their "structural" or "reduced-form" nature. The latter is not particularly useful for the issuance of new warrants. Therefore, the structural approach is the more appropriate when a new warrant issue has to be priced, for example in the context of a management package.

## 3.1 Warrant as a call option on the value of the firm

### 3.1.1 The Black-Scholes model applied to the value of the firm

This model belongs to the structural approach because the exogenous variables are the value of the firm and its volatility and, consequently, the value of the stocks and warrants of the firm are endogenous to the model.

In table 1, row (2.b) displays the value of the warrant at maturity, $T$:

$$W_T = \frac{1}{M+N}(\tilde{V}_T - NK)^+ \quad (2)$$

$$= \frac{N}{M+N}(\tilde{S}_T^* - K)^+ \quad (3)$$

In the Black and Scholes (1973) framework, the following assumptions are made: the financial markets are perfect and complete and transactions takes place in continuous time. There is no arbitrage opportunity. The risk-free interest rate is constant, the stock does not pay a dividend and its price follows a Geometric Brownian Motion (GBM hereafter) with constant drift and volatility.

Here, this model is applied not to the stock price but to the firm's asset value, $\tilde{V}_t$, whose dynamics is described by a GBM. It follows from all the previous assumptions that a unique equivalent martingale measure (commonly referred-to the "risk neutral probability") exists under which the stochastic process for $V$ becomes:

$$\frac{dV_t}{V_t} = rdt + \sigma_V d\tilde{W}_t$$

where

$\tilde{W}_t$ is a standard Brownian motion with respect to the equivalent martingale measure, $\mathbb{Q}$.

$\sigma_V$ is the volatility of the instantaneous return of $V_t/N = S_t + (M/N)\, W_t$,

Thus, the price of any assets in this complete market is simply the discounted expected value of its future payoff under $\mathbb{Q}$.

Then, the following formula for the warrant price is obtained:

$$\begin{aligned} W_t\left(V_t, \sigma_V; \tau, K, r\right) &= \frac{1}{M+N}\left(V_t \Phi(d_1) - NKe^{-r\tau}\Phi(d_1)\right) \\ &= \frac{N}{M+N}\left[\left(S_t + \frac{M}{N}W_t\right)\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)\right] \\ &= \frac{N}{M+N}C_t^{BS}\left(V_t/N, \sigma_V; \tau, K, r\right) \end{aligned} \quad (4)$$

with

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_t + \frac{M}{N}W_t}{K}\right) + (r + \sigma_V^2/2)\tau}{\sigma_V\sqrt{\tau}} \\ d_2 &= d_1 - \sigma_V\sqrt{\tau} \end{aligned}$$

where

$\Phi(.)$ is the cumulative distribution function of the standard normal distribution.

Expression (4) for the value of the warrant corresponds to the standard Black-Scholes formula in which:

- the value of the equity per share $V_t/N = S_t + (M/N)\, W_t$ replaces the stock price $S_t$,

- the volatility $\sigma_V$ is the volatility of the equity of the firm (*i.e.*, shares of stock plus warrants) instead of just the stock's volatility, $\sigma_S$,

- the formula is multiplied by $N/\left(M+N\right)$.

This valuation formula is known in the warrant literature as the classical warrant formula or the "correct warrant valuation model" (Veld 2003). The main obstacle to the application of this formula is that it depends on the value of the firm, $V_t$, and on its instantaneous volatility, $\sigma_V$, which are not observable. Moreover, the value of a warrant depends on its own value through the value of the firm (i.e., $V_t = NS_t + MW_t$) on the right hand side of expression (4). This simply requires the use of a numerical root finding routine.

As shown by Crouhy and Galai (1991a) and Bensoussan, Crouhy and Galai (1994, 1995a and 1995b), the fact that the model starts from the exogenously given stochastic process of the value of the firm, $V$, which is not observable and not from that of the stock price, $S$, implies that the stock price itself must be analyzed as a contingent claim on the firm's asset, $V$. This in turn implies that the stochastic process for the stock price is no longer stationary which means in particular that its volatility is not constant over time. This comes from the impact of warrants issuance on the capital structure of the firm. As mentioned by Bensoussan *et al.* (1995): "By issuing warrants, part of the total asset risk is shifted from the stockholders to the warrant-holders, and the degree of this risk-sharing changes continuously as a function of the asset value and the time-to-maturity of the warrants."

### 3.1.2 The Schulz and Trautmann (1994), Bensoussan, Crouhy and Galai (1995a) and Ukhov (2004) model

The formula (4) is not really useful in practical applications because it relies on non observable variables, $V_t$ and $\sigma_V$. It's why, based on the previous model, the authors cited above in the title of the section introduce a method to express the warrant value as a function of only observable variables, namely the current stock price and the volatility of the stock return. Therefore, we must find a way to express $S_t$ and $\sigma_S$ as function of $V_t$ and $\sigma_V$: $S_t \equiv S(V_t, \sigma_V)$ and $\sigma_S \equiv \sigma_S(S(V_t, \sigma_V))$.

**1.** We already have the relationship between $S_t$ and $(V_t, \sigma_V)$ (see Table 1, line 2.d.):

$$S_t = V_t/N - (M/N)\, W_t(V_t, \sigma_V) \tag{5}$$

**2.** To find the relationship between $\sigma_S$ and $(V_t, \sigma_V)$, we start with the observation that, for a \$1 change in the firm's value ($\Delta_V = 1$), we have from (5):

$$\begin{aligned} \Delta_S &= \frac{1}{N} - \frac{M}{N}\Delta_W \\ &= \frac{1}{N} - \frac{M}{N(N+M)}\Phi(d_1) \end{aligned}$$

If we analyze the stock as a contingent claim on the value of the firm, the expression of the elasticity of the stock price with respect to the value of the firm is given by: $\Omega_{S,V} = (\partial S/\partial V) \cdot (V_t/S_t) = \Delta_S (V_t/S_t)$. A standard result from stock option theory states that the the standard deviation of the rate of return of a call equals its elasticity times its underlying stock volatility (see Cox and Rubinstein, 1985). This translates here into the following[8]:

$$\sigma_S(S(V_t, \sigma_V)) = \Omega_{S,V} \cdot \sigma_V \tag{6}$$

Expression (6) shows that the volatility of the stock is not constant, it is a function of the value of the firm and of time. Furthermore, because stock and warrant returns are perfectly positively correlated, the volatility of the assets of the firm is simply the weighted average of the stock's volatility and of the warrant's volatility:

[8] The detailed derivation can be found in the three papers already cited. The most rigorous proof can be found in Bensoussan, Crouhy and Galai (1995). Crouhy and Galai (1991) were the first to prove that stock volatility is no longer stationary if firm's equity is analyzed as a contingent claim on the firm's assets.

$$\sigma_V = \sigma_S \frac{N.S_t}{V_t} + \sigma_W \frac{M.W_t}{V_t} \tag{7}$$

If the firm's asset risk is assumed to be constant over time, $\sigma_V = cste$, and because the warrant is riskier than the stock, relationship (7) implies that the volatility of the stock is less than the volatility of the firm's assets. Note that when a firm is also financed by debt, the volatility of the stock may be higher than the volatility of the firm's assets. As recalled by Bensoussan, Crouhy and Galai (1994 and 1995b): "While debt causes the risk of equity to increase relative to the volatility of the firm, the issuance of diluting securities like warrants has the opposite effect". Moreover, expression (7) also implies that the stock's volatility varies over time. In fact, as a function of $S_t$ and therefore $V_t$, the stock's volatility is stochastic.

Given $S_t$ and $\sigma_S$, the price of the warrant is obtained in two steps:

**1.** First, the following system of nonlinear equations for $(V_t, \sigma_V)$ must be solved numerically:

$$\begin{cases} S_t = V_t/N - (M/N)W_t(V_t, \sigma_V) \\ \sigma_S = \Delta_S(V_t/S_t)\sigma_V \end{cases} \tag{8}$$

**2.** Second, the warrant price is given by:

$$W_t\left(V_t, \sigma_V\right) = \frac{V_t - S_t N}{M}$$

Ukhov (2004) proves that this system always admits a solution and Pechtl and Trautmann (2003) prove its uniqueness.

## 3.2 Warrant as a call option on the stock of an identical all-equity firm

The value of a call option on the stock of an identical fictitious all-equity firm at maturity, $C_T$, is :

$$C_T = (\tilde{S}_T^* - K)^+$$

Relationship (3) allows to write:

$$W_T = \frac{N}{M+N} C_T$$

Thus by virtue of the principle of no arbitrage opportunity:

$$W_t = \frac{N}{M+N} C_t^{BS}(S_t^*, \sigma_{S^*}; \tau, K, r), \ \forall t < T$$

The main limitation in applying this approach is that it is unlikely to find an identical all-equity firm to the one that issues warrants.

## 3.3 Warrant as a call option on the stock of the firm

This model belongs to the reduced-form approach. In this case, the warrants can be simply valued as a call options on the stock of the firm:

$$W_t = C_t^{BS}(S_{t,}\sigma_S; \tau, K, r), \ \forall t \in [0, T]$$

The use of a Black-Scholes model with a GBM for the stock price dynamics seems to contradict what was said in section 3.1.1 on the variation over time of stock volatility in a structural model. But, Jarrow and Trautmann (2011) showed that a reduced-form model could be compatible with a GBM, in particular with a constant volatility of the stock. Their argument rely on the difference of the information set[9] in a structural and a reduced form model.

For this formula to be valid, the share price must already reflect the issue of warrants and therefore the dilution that it entails. As pointed out by Handley (2002), when a warrant is issued for cash and the proceeds reinvested in a scale expansion of the firm's assets, the stock price should not adjust. On the contrary, if the proceeds are paid to shareholders in the form of dividends, the share price should adjust downward by an amount close to that of the warrant price.

Thus either the warrants are already outstanding or the issue is about to occur and, in an efficient market, this information is already incorporated into the stock price. Note that in the case considered here, it is not necessary to explicitly adjust the call formula by a dilution factor since the dilution is already implicitly reflected in the stock price.

If the warrant is valued as a function of the stock price before the market has this information, then an explicit adjustment for dilution must be made (Handley, 2002).

# 4 Approximation quality of a simple Black-Scholes model

Because only $S_t$ and $\sigma_S$ are observable, most practitioners base their Black-Scholes type warrant pricing model on $S_t$ and a constant $\sigma_S$. But instead of solving the system of equations (8), they simply value a warrant as a call option on the stock of the firm as in section 3.3. Then, it is interesting to evaluate the quality of such an approximation compared to the correct valuation model of section 3.1.1. Before that, we perform numerical simulations to highlight the non-stationarity of the stock's volatility.

## 4.1 Stock's Volatility

As first shown by Crouhy and Galai (1991a), in a structural model like the one in section 3.1.1 the stock's volatility is a function of $S$ and $t$ given by $\sigma_S \equiv \sigma_S\left(S_t\left(V_t, \sigma_V\right), t\right)$. Then, it would be interesting to analyze how stock's volatility changes with $S$ and $t$. To this end, we start from given $V$ and $\sigma_V$ and derives the corresponding $S$ and $\sigma_S$. Using the approach of Schulz and Trautmann (1994), we use the system of equations (8) with the following parameter values: $\sigma_V = 0.3$, $r = 0.01$, $M = N = 1$, $K = V/N = 100$ (we consider a firm's value of $V = 100$ as our base case). We perform simulations for a range of values of $V$ from 0 to 500 and $\tau$ from 0 to 10. For each pair $(V, \tau)$, we compute the corresponding $(S, \sigma_S)$. The results are shown in Figure (1).

[9] By information set, they refer to the technical definition of the filtration of the probability space.

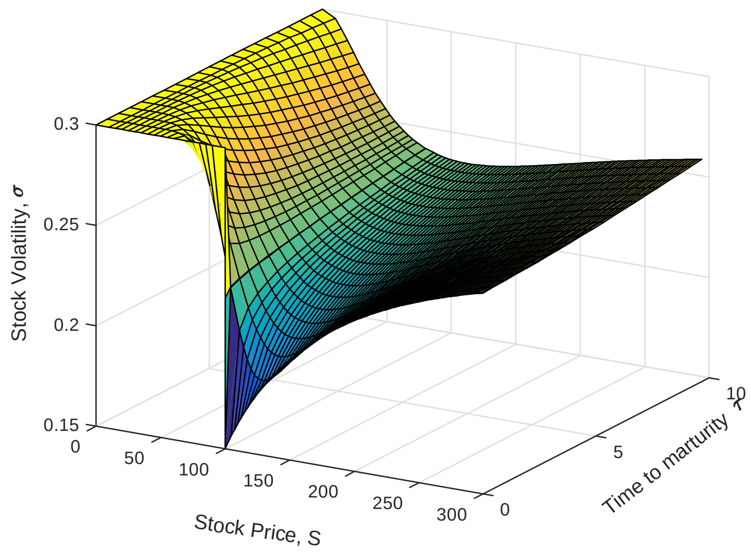


Figure 1: Stock Volatility ($N = 1$)

For $\tau \neq 0$, stock's volatility first decreases with $S$, reaches a minimum then becomes increasing and converges toward $\sigma_V$ as $S$ becomes very large. As the expiration date approaches ($i.e.$, $\tau \to 0$), stock's volatility behavior becomes unstable around $V/N = S = K$. This is due to the behavior of the option's delta which converges either to 1 for $V/N > K$, 1/2 for $V/N = K$ and 0 for $V/N < K$. The volatility of the stock[10] is equal to $\sigma_V$ as long as the option is out-of-the-money, then it jumps to $\sigma_S = \frac{3}{4}\sigma_V$ at $S = K$ and reaches a minimum value equal to $\sigma_S = \frac{1}{2}\sigma_V$ for $S \to K^+$ after which it increases and converges to $\sigma_V$.

We also display on Figure (2) the effect of a lower level of dilution by considering $M = 1$, $N = 10$ with $K = V/N = 10$.

[10] By using the expression for the stock's volatility, $\sigma_S = \frac{V}{S}\left[\frac{1}{N} - \frac{M}{N(N+M)}\Phi\left(d_1\right)\right]\sigma_V$, and the fact that

$$\lim_{\tau \to 0^+} \Phi(d_1) = \begin{cases} 1 \text{ if } V/N > K \\ \frac{1}{2} \text{ if } V/N = K \\ 0 \text{ if } V/N < K \end{cases}$$

we obtain the desired result

$$\lim_{\tau \to 0^+} \sigma_S = \begin{cases} \frac{1}{2}\sigma_V \text{ if } V/N > K \\ \frac{3}{4}\sigma_V \text{ if } V/N = K \\ \sigma_V \text{ if } V/N < K \end{cases}$$

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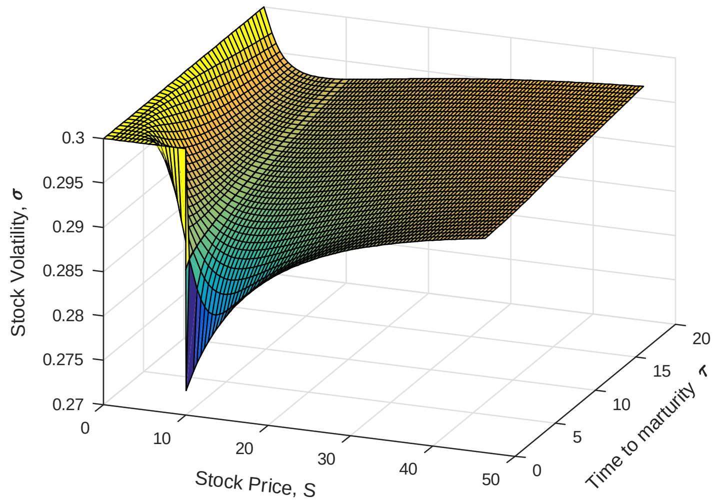


Figure 2: Stock Volatility ($N = 10$)

The general shape of stock's volatility surface is the same as the one with $N = 1$. As expected, the only difference is that the minimum stock's volatility is now much higher and is equal to 27.27%. This comes from the much lower dilution. Likewise, the range of variation of the stock's volatility is narrower. The fact that the dilution is 10 times lower than in the previous case means that the share of the value of the warrant in the total value of the firm is much lower. Thus, there is less of the risk of the firm that is transferred from stockholders to warrant holders, which explains the higher level of stock's volatility.

The previous discussion remains exactly the same for other values of the firm's volatility.

As a function of time to maturity, the stocks's volatility is decreasing as long as $S_t < K = V/N$. For $S_t \geq K = V/N$, it becomes an increasing almost "linear" function of time to maturity. Let's go into more detail.

The behavior of the volatility of the stock with respect to the time to maturity for a given $(V, \sigma_V)$ is driven both by the stock's price $S$ and by its delta $\Delta_S$. *Ceteris paribus*, the stock price increases when the expiration date approaches[11] and the delta behavior depends on the moneyness of the warrant. Taking into account both effects, we can conclude that the volatility of the stock decreases almost linearly as we get closer to maturity and for deep in-the-money warrant.

As the warrant get closer to the money, the relationship is still decreasing but "less linear".

For out-of-the-money warrant, the stock's volatility increases when the warrant approaches the expiration date. Note that for the slightly in-the-money warrant, the range of variation in the volatility of stock is the greatest and is about $\frac{1}{2}\sigma_V$ for very long term warrants.

An estimate of the standard deviations of the time series of the stock's rate of return instead of the volatility of the firm's asset will lead to an underestimation of the latter and therefore of the

[11] Because of the warrant theta.

price of the warrant. This underestimation is all the more important as the warrant is close to the money and near maturity[12].

### 4.2 Black-Scholes approximation

It is tempting for practitioners to value a warrant by simply applying the Black-Scholes formula without any adjustments. Formally, this is correct if the stock's price already reflects the information about the warrants issuance. But if we try to evaluate the issue price of a warrant before the market has this information, then that is no longer correct.

Another practice would be to make the wrong adjustment as highlighted by Crouhy and Galai (1991b). For example, the model of section (3.2) can be applied, but on the firm with warrants whereas it is only a correct model for an identical pure-equity firm. By doing this, the additional dilution factor in the expression $W = \left(\frac{N}{N+M}\right) C$ will biased the valuation of the warrant.

In any case, we will analyze the accuracy of the Black-Scholes valuation of the price of a warrant compared to the correct valuation of a warrant for which we use information available to the market, $(S, \sigma_S)$.

In this section unlike the previous one, given $(S, \sigma_S)$, we numerically solve the system of equations (8) to obtain $(V, \sigma_V)$ and then the warrant price, $W$. Proceeding in this way for each value of the parameters, we are led to compare firms with different asset values, which does not seem very consistent. But remember that our objective here is to compare the accuracy of the Black-Scholes approximation for a given set of parameter values. Moreover, we still rely on the same parameter values as those in the previous sections[13]. Specifically, we assess the accuracy of the approximation by calculating the following error term:

$$E(S_t,\sigma_S;\tau,K,r) = \frac{C_t^{BS}(S_t,\sigma_S;\tau,K,r)}{(N/(N+M))C_t^{BS}\left(V_t/N,\sigma_V;\tau,K,r\right)} - 1 \tag{9}$$

First of all, it is worth noting that most of the time the error term is positive, which means that the Black-Scholes approximation overestimates the true value of the warrant. By inspecting expression (9), we see that the sign of $E(S_t,\sigma_S;\tau,K,r)$ depends on two effects working in opposite directions:

- first, given that $\sigma_S \leq \sigma_V$ and $S \leq V/N$ : $C_t^{BS}\left(V_t/N,\sigma_V;\tau,K,r\right) > C_t^{BS}(S_t,\sigma_S;\tau,K,r)$,
- second, $N/(N+M) < 1$.

We simulate the error term for the following set of parameter values: $r = 1\%$, $S \in [50;150]$, $\sigma_S \in [20\%;100\%]$, $M/N \in [0.1;1]$ and $\tau \in \{0.5,5,10\}$. The Black-Scholes approximation works very well for at-the-money and in-the-money warrant. The percentage price difference is in the range $[-1.41\%, 3.61\%]$ for $S \in [100;150]$.[14] As expected, the approximation is all the more precise

[12] This remark does not apply to the case where an estimate of $\sigma_S$ is used to find $\sigma_V$ by solving the system of equations (8).

[13] Because the current level of interest rates in the markets is very low, we use $r = 1\%$. To assess the effect of a higher interest rate, we consider in the next section the effect of a $r = 10\%$ on the approximation error committed.

[14] The lower bound of $-1.41\%$ is obtained for $M/N = 100\%$, $\sigma_S = 20\%$, $\tau = 0.5$ and $S = 107$. The upper bound of 3.61% is obtained for $M/N = 100\%$, $\sigma_S = 88\%$, $\tau = 5$ and $S = 100$.

the lower the dilution. For out-of-the-money warrant, the range of the approximation error is wider and is equal to [0.022%, 100.11%] for $S \in [50; 99]$.[15] Note that in this case, the approximation error is always positive. This is only true because the interest rate is 1%. For example, if the interest rate is 10%, negative approximation errors occur as soon as $S \geq 96$ and $\tau = 0.5$.

We display on figures 3, 4 and 5 the approximation error as a function of both the dilution ($M/N \in [0.1; 1]$) and the stock volatility $\sigma_S \in [20\%; 100\%]$. On each figure, we draw three surfaces for the time to maturity $\tau \in \{0.5, 5, 10\}$.

The analysis of the effect of the stock's volatility on this price difference is complex as can be seen on Figures 3, 4 and 5. The quality of the approximation depends also on the dilution, on the time to maturity and on the moneyness of the warrant. We will distinguish three cases according to the moneyness of the warrant:

**1.** $S = 50$, $\tau = 10$ and $K = 100$: for a high dilution of 100% and a low volatility level of 20%, the price difference is very high at 18.82%. This happen for a relatively high level of warrant price 3.25 and a corresponding call value of 3.86. For this dilution of 100%, the price difference is first decreasing then slightly increasing and finally decreasing again with $\sigma_S$ as can be seen on Figure 3. It is still equal to 4.3% for $\sigma_S = 60\%$. When $\tau = 5$, the behavior is the same but the highest difference is now equal to 47.89%, but for a much smaller warrant value of 0.69 and a corresponding call value of 1.02. Although, the price difference looks very important for $\tau = 0.5$ (about 100% !), it has no economic meaning because the prices of both the Call and the warrant are negligible.

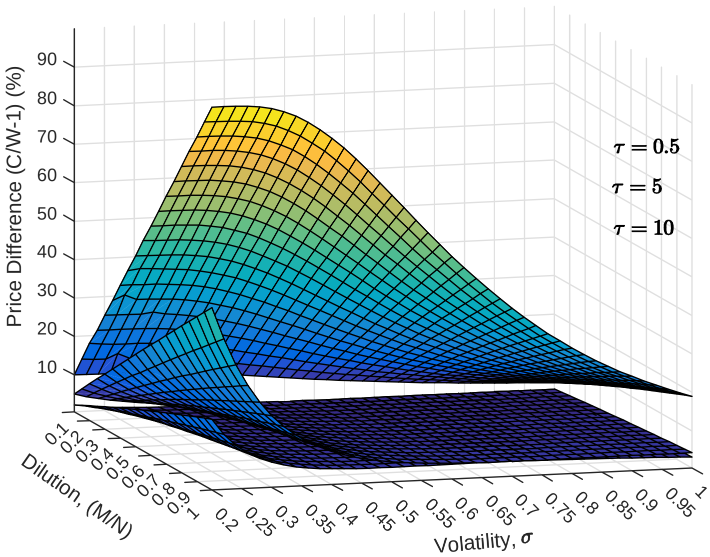


Figure 3: Price Difference (%) for $S = 50$ and $K = 100$.

Notes: This figure shows the approximation error for a deep out-of-the-money Warrant ($S = 50$ and $K = 100$) as a function of the volatility, $\sigma$, and of the dilution, $M/N$, for three maturities, $\tau = 0.5$, 5 and 10 years.

[15] The lower bound of 0.022% is obtained for $M/N = 10\%$, $\sigma_S = 20\%$, $\tau = 5$ and $S = 99$. The upper bound of 100.11% is obtained for $M/N = 100\%$, $\sigma_S = 22\%$, $\tau = 0.5$ and $S = 56$.

Thus, for out-of-the money warrants, the use of a simple Black-Scholes formula instead of the correct valuation formula for warrants leads to an overestimation of the warrant price which can be very important even for time to maturity equal to 5 and 10 years. In this case, only overestimates occur.

**2.** $S = 100$, $\tau = 10$ and $K = 100$ (see Figure 4)[16]: for a dilution of 100% and a low volatility level $\sigma_S = 20\%$, the price difference is much lower than in the previous case at 1.25%, then it increases and reaches a maximum at 3.55% for a stock's volatility of 65% and then decreases. When $\tau = 5$, we have the same behavior but the price difference is lower at 0.71% (for $M/N = 1$ ) with a maximum of 3.613% for $\sigma_S = 88\%$. For $\tau = 0.5$. the price difference can be negative for low dilution and low volatility leading to an underestimation of the true value by use of the simple Black-Scholes formula.

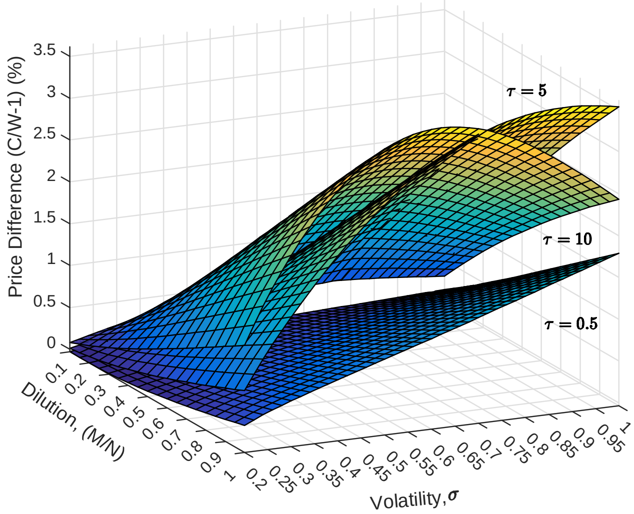


Figure 4: Price Difference (%) for $S = 100$ and $K = 100$.

Notes: This figure shows the approximation error for at-the-money Warrant ($S = 100$ and $K = 100$) as a function of the volatility, $\sigma$, and of the dilution, $M/N$, for three maturities, $\tau = 0.5$, 5 and 10 years.

**3.** The behavior for in-the-money warrants is the same except that the range of variation is narrower as can be seen on Figure 5. But here, we must distinguish between in-the-money warrants and deep in-the-money warrants. Indeed for in-the-money and near maturity warrants, the price difference can be negative for a large range of dilution and volatility values.

### 4.3 Approximation Error at issuance in a low interest rates environment

The current low level of interest rates can have a substantial impact on many economic variables. For example, Escobar *et al.* (2019) highlight the challenge facing insurance companies. Amédée

[16]Note that the order of the three surfaces according to $\tau$ is not the same as in Figure(3).

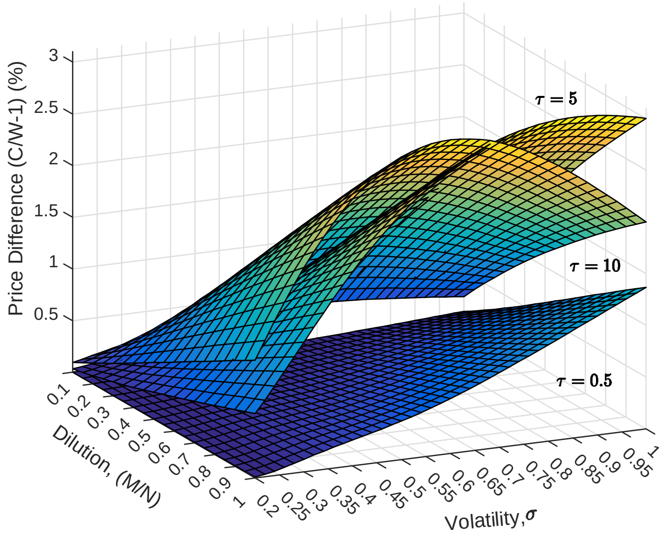


Figure 5: Price Difference (%) for $S = 150$ and $K = 100$.

Notes: This figure shows the approximation error for deep in-the-money Warrant ($S = 150$ and $K = 100$) as a function of the volatility, $\sigma$, and of the dilution, $M/N$, for three maturities, $\tau = 0.5$, 5 and 10 years.

*et al. (2019)* show in a dynamic portfolio optimization context that low interest rates lead to an increase in the real estate allocation. It also has an impact on the accuracy of the approximation of the price of a warrant by a simple Black-Scholes model.

Typically, when a warrant is about to be issued as, for example, part of a management package[17], it is very out-of-the-money with a fairly long maturity (typically, $T = 5$ years) and it is very likely that the volatility of the stock is moderate to high and is likely to be estimated with high level of uncertainty. The approximation error can therefore be substantial as shown in Figure 3, even with a moderate level of dilution. Moreover in this particular case, the pricing error is very sensitive to the stock's volatility. This means that an error in estimating volatility can have a significant impact on the pricing error of the warrant price approximated by the simple Black-Scholes value. This features is reinforced by the low level of interest rates as can be seen in the Figures 6.

It is worth noting that in the current low interest rates environment, the pricing error problem is more pronounced. When the first simulation studies of the pricing error were undertaken some thirty years ago, interest rates were high and most authors ran their simulations with a typical interest rate of 10%. Today, typical interest rates, even for a long maturity, are around 1%. And as we can see in the Figures 6, this matters. First, the level of the pricing error is much higher, *ceteris paribus*, with an interest rate of 1% than with a rate of 10%. Second, the range of pricing error

[17] The term management package refers to the remuneration systems for executives, particularly in LBO operations. The management directly invests part of its assets in the company through warrants, convertible bonds or shares. The idea is to incentivize managers to succeed and therefore to set up asymmetric remuneration schemes. Stock warrants are therefore the perfect tool for this purpose.

for volatilities between 20% and 60% is much higher with low interest rates. And this remains true whether the dilution is 10% or 50%.

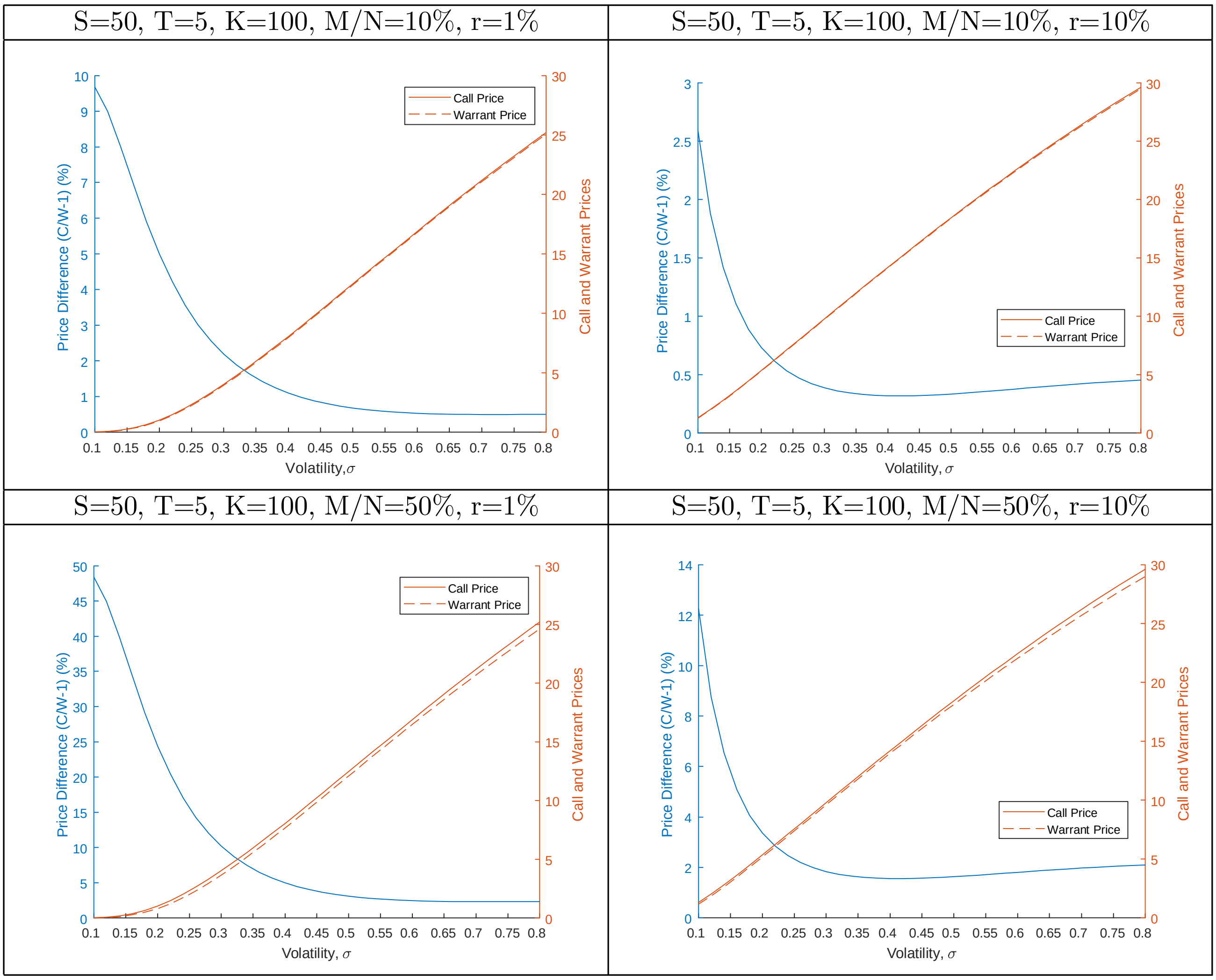


Figure 6: Pricing Error, Call and Warrant Prices as a function of volatility

Table 2 (obtained with $M/N = 50\%$) gives a more precise idea of the magnitudes. By looking at it for $r = 1\%$ and $\sigma_S = 30\%$, the Black-Scholes formula overestimates the warrant price by 10.2%. For the same volatility but with $r = 10\%$, the overestimation is now only 1.83%. For a stock volatility of 50%, the Black-Scholes formula overestimates the warrant price by 3.11% for $r = 1\%$ and by only 1.63% for $r = 10\%$. It's only for very high stock's volatility, such as $\sigma_S = 80\%$ and above, that the impact of an interest rate difference is very small. We can verify that for a dilution between 10% and 100% the impact of an interest rate difference is always significant up to a volatility level of around 60%.

To better understand the effect of the interest rate on the approximation error, we display in the Figures 7 the pricing error as a function of the level of the interest rate for two values of the volatility ($\sigma = 20\%$ and 40%) and for two values of the dilution (M/N=10% and 50%). It is inversely related

Table 2: Pricing Error, Call and Warrant Prices

| $\sigma_S$(%) | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
|---|---|---|---|---|---|---|---|
| | | | | **r=1%** | | | |
| Call | 1.02 | 4.02 | 8.06 | 12.47 | 16.90 | 21.18 | 25.20 |
| Warrant | 0.82 | 3.65 | 7.68 | 12.09 | 16.49 | 20.70 | 24.62 |
| C/W-1 (%) | 24.43 | 10.20 | 4.99 | 3.11 | 2.47 | 2.33 | 2.34 |
| | | | | **r=10%** | | | |
| Call | 5.34 | 9.78 | 14.19 | 18.44 | 22.45 | 26.19 | 29.62 |
| Warrant | 5.17 | 9.60 | 13.97 | 18.14 | 22.05 | 25.68 | 29.01 |
| C/W-1 (%) | 3.35 | 1.83 | 1.56 | 1.63 | 1.80 | 1.97 | 2.09 |

Notes: This Table displays the approximation error, the Call and Warrant prices for $M/N = 50\%$, $r = 1\%$ and 10% and for stock's volatilities between 20% and 80%.

to the level of the interest rate and is very important for a dilution of 50% and a volatility of 20%. Note also the strong sensitivity of the pricing error to the interest rate, as shown by the steepness of the blue curve.

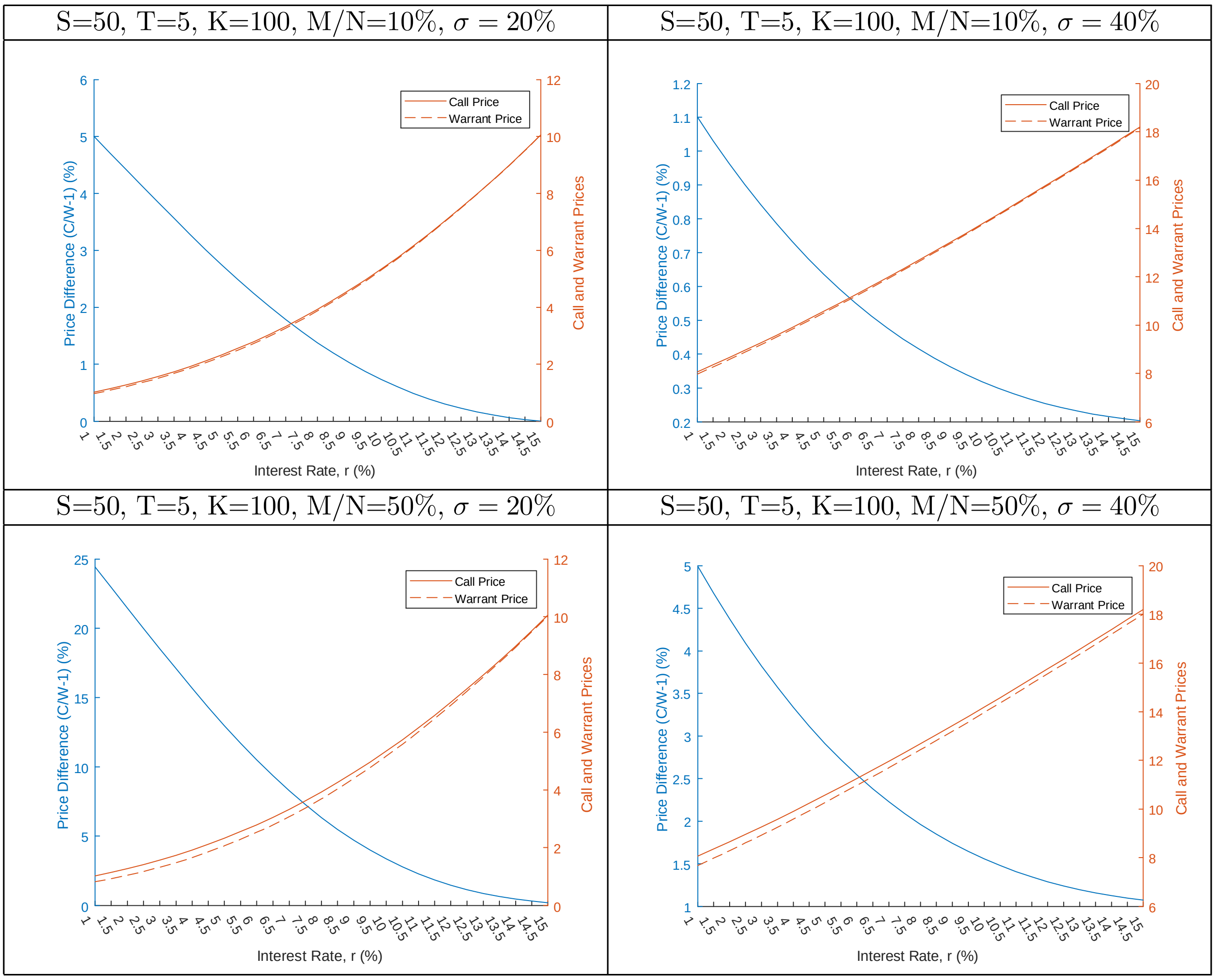


Figure 7: Pricing Error, Call and Warrant Prices as a function of the interest rate

In Figure 8, we can see simultaneously the effect of volatility and interest rate on the pricing error. It highlights the sensitivity of the pricing error to volatility in a low interest rate environment.

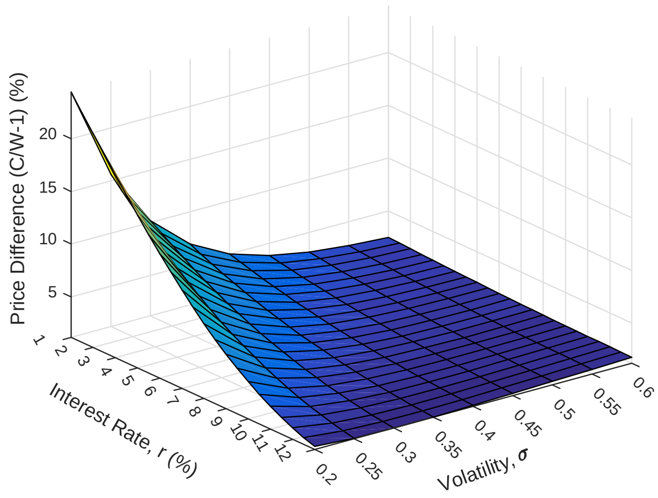


Figure 8: Pricing Error as a function of the interest rate and of the volatility

Notes: This figure shows the approximation error for out-of-the-money Warrant ($S = 50$ and $K = 100$) as a function of the interest rate, $r$, and of the volatility, $\sigma$, for a time to maturity $\tau = 5$.

## 5 Concluding Remarks

The objective of this paper was first to recall the rich nature of warrants compared to traded options. This is so because these securities modify the capital structure of firms. Even in the simple case where we assume that firms do not finance by debt, the valuation of warrants is complex, and several approaches are possible as recalled in the text. We then focused on the quality of the approximation of the issue price of a warrant by a simple Black-Scholes call formula, a common practice of practitioners. In particular, we have shown that the approximation error is higher and more sensitive to the stocks's volatility parameter in the current low interest rate environment.

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